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Magnetohydrodynamic flow in channels of variable cross-section with strong transverse magnetic fields

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This paper is an analysis of the steady incompressible, two-dimensional flow of conducting fluids through ducts of arbitrarily varying cross-section under the action of a strong, uniform, transverse, magnetic field. More precisely, the flow is such that the velocity is given by $\mathbf{u} = (u_x(\tilde{x}, \tilde{y}), u_y(\tilde{x}, \tilde{y}), 0)$, the position of the duct walls by $\tilde{y} = f_t(\tilde{x}), f_b(\tilde{x})$ and $\tilde{z} = \pm b$, where $b \gg f_t - f_b$, and the magnetic field by $\mathbf{B}_0 = (0, B_0, 0)$. It is assumed that the magnetic field is strong enough to satisfy the conditions that the interaction parameter, $N(= M^2/R) \gg 1$, where M is the Hartmann number and R is the Reynolds number, and also that $M \gg 1$ and $R_m \ll 1$, where R_m is the magnetic Reynolds number.

We examine the flow in three separate regions:

(i) the 'core' region in which the pressure gradient is balanced by electromagnetic forces;

(ii) Hartmann boundary layers where electromagnetic forces are balanced by viscous forces;

(iii) thin layers parallel to the magnetic field in which electromagnetic forces, inertial forces, and the pressure gradient balance each other. These layers which have thickness $O(N^{-\frac{1}{3}})$ occur where the slope of the duct wall changes abruptly.

By expanding the solution as a series in descending powers of N we calculate the velocity distribution for regions (i) and (ii) for finite values of N attainable in the laboratory.

1. Introduction

In this paper we consider the effect of a strong uniform magnetic field

$$\mathbf{B}_0 = (0, B_0, 0)$$

on steady, two-dimensional flows, whose velocities are given by

$$\mathbf{u} = (u_x(\tilde{x}, \tilde{y}), \quad u_y(\tilde{x}, \tilde{y}), 0),$$

through ducts with walls at

$$\tilde{y} = f_t(\tilde{x}), f_b(\tilde{x}) \quad \text{and} \quad \tilde{z} = \pm b,$$

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where we assume $b \gg f_t - f_b$. (The effect of the walls at $\tilde{z} = \pm b$ on the flow is considered negligible.) Our analysis also enables us to examine the flow over a two-dimensional body placed in such a duct (see figure 1). We assume that the fluid is incompressible, that it has uniform properties, and that its conductivity is such that the magnetic Reynolds number (R_m) is much less than unity.

There are two main reasons for studying these flows, apart from their intrinsic theoretical interest. First, although many MHD devices involve flow in converging or diverging ducts, there have been apparently few theoretical attempts to analyse them. Most of the existing theory for laminar, incompressible MHD duct flow has been developed for cases where the cross-section is uniform (for a list of references see Hunt 1965). Axford (1961) examined the flow in ducts with straight diverging walls, but assumed the magnetic field to decrease inversely as the distance from a source point. Secondly, there is a great need for experimental work to test MHD theory and so it is important that the theory be in such a form that it can be tested in the laboratory. The situation examined in this paper is one which can be reproduced experimentally using liquid metals and it is hoped that the formulae which are set out here will be of use in this connexion.

Ludford (1961) and Ludford & Singh (1963) have developed much of the existing theory for external flows in transverse magnetic fields for two and three dimensions. They assumed the magnetic field to be strong and the conductivity weak enough so that the induced field could be ignored. More precisely, they required that the interaction parameter $N(=\sigma B_0^2 a/\rho U_0) \gg 1$, and that the magnetic Reynolds number $R_m(=\mu\sigma U_0 a) \ll 1$, where B_0, σ, ρ, μ are the flux density of the imposed transverse magnetic field, the fluid conductivity, density, and the magnetic permeability, respectively. U_0 and a are the characteristic velocity and length.

Making this approximation for two-dimensional flow, Ludford (1961) found that in the 'inner' region near the body the inertial terms in the momentum equation are negligible compared to the electromagnetic and pressure forces, except in singular zones at the front and rear of the bodies. In the 'outer' region sufficiently far in the field or y direction from the body, he found that inertial forces again became important. He discussed their effect by compressing the y co-ordinate to retain the influence of inertia and found expressions for lift and drag. The singularities in the inner region were left, and the question of how these may affect the flow was not resolved.

We apply Ludford's approximations to internal flows with due regard to viscous effects at the walls. The entire 'inviscid' core flow then corresponds to his 'inner' region. As in external flows, singular zones arise in the core flow whenever the duct wall curvature is $O(N)$, and it is this difficulty which is resolved in our treatment of the core. It is found that the singular zones are the limit as $N \to \infty$ of thin layers of thickness $O(N^{-\frac{1}{3}})$ in which inertial forces, electromagnetic forces and the pressure gradient are balanced. When the longitudinal or x co-ordinate is stretched to reveal this, Ludford's (1961) principal equation (his equation (13)) is reproduced. In solving this equation we use a simple transform method for internal flows and find that we can use the fundamental solution which Ludford developed in his paper to construct a uniform approximation to his external flow problem.

Unlike a wind tunnel, a duct for investigating MHD flow over bodies has to be placed in a magnet whose gap is usually small. Consequently the duct size is severely limited and for flow over a practical size of body wall effects are important in the inviscid regions as well as the boundary layers. We examine the effects of the wall on the inviscid regions using Ludford's approximation and we examine the boundary layers by assuming that the Hartmann number $M\{=B_0 a(\sigma/\eta)^{\frac{1}{2}}\} \gg 1$. Here η is the viscosity. An interesting feature of this calculation is that higher order approximations to the Hartmann layer can easily be found. This is possible because of the simplicity of the core flow solution (away from singular zones), for which an expansion in inverse powers of N may easily be found.

The approximation used by Ludford has also been used, very successfully, by Bornhorst (1965) to calculate the effect of a magnetic field on the free surface of a mercury flow when $N \gg 1$. The fact that the theory accurately predicted the free surface profiles found experimentally demonstrates the usefulness of the approximation. It is worth observing that, in general, it is not difficult to devise laboratory experiments which satisfy our criteria that $N \gg 1$, $R_m \ll 1$ and $M \gg 1$, while having the Reynolds number large enough for accurate readings of pressure, velocity, etc., to be taken. On the other hand our criteria are not satisfied by the flows in practical MHD devices at the moment (e.g. in the biggest MHD generators N is only $O(1)$). However, as their size and their field strength increase, so that N increases, our approximate methods may become increasingly useful in examining the flows in MHD pumps, generators, etc.

2. Statement of the problem

The magnetohydrodynamic (MHD) equations for steady, incompressible flow when the fluid properties are constant are:

$$\rho(\mathbf{u}.\nabla)\mathbf{u} = -\nabla\tilde{p}+\mathbf{j}\times\mathbf{B}+\eta\nabla^2\mathbf{u}, \quad (2.1)$$

$$\nabla.\mathbf{u} = 0, \quad (2.2)$$

$$\mathbf{j} = \sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}), \quad (2.3)$$

$$\nabla\times\mathbf{E} = 0, \quad (2.4)$$

$$\mathbf{j} = (1/\mu)\nabla\times\mathbf{B}, \quad (2.5)$$

$$\nabla.\mathbf{B} = 0, \quad (2.6)$$

where $\mathbf{u}$, $\tilde{p}$, $\mathbf{j}$, $\mathbf{B}$, $\mathbf{E}$ are velocity, pressure, current density, magnetic flux density and electric field respectively. When $R_m \ll 1$, we can ignore the induced magnetic field due to $\mathbf{j}$ and assume that, in (2.1) and (2.3),

$$\mathbf{B} = \mathbf{B}_0,$$

where $\mathbf{B}_0$ is the imposed magnetic field.

If now we consider a two-dimensional flow in the $(\tilde{x}, \tilde{y})$-plane, such that

$$u_z = \partial/\partial\tilde{z} = 0,$$

then $$\partial E_x/\partial\tilde{z} = \partial E_y/\partial\tilde{z} = 0,$$

and hence from (2.4) $$\partial E_z/\partial\tilde{x} = \partial E_z/\partial\tilde{y} = 0.$$

Whether the walls at $\tilde{y} = f_t(\tilde{x})), f_b(\tilde{x})$ are conducting or not, it may be shown that $E_x = E_y = 0$ provided there are no current sources or sinks along these walls. (If the electrical boundary conditions on the walls at $\tilde{z} = \pm b$ vary rapidly in the x-direction then it follows that $\partial/\partial z \neq 0$ and $E_x \neq 0$; thus the applicability of the basic assumptions to real flows must always be carefully checked. We discuss this point further in the conclusion.) If the magnetic field $\mathbf{B}_0$ lies in the y-direction and if we reduce the parameters to a non-dimensional form in terms of Q, the total flow rate through the duct per unit depth, B_0, and a, a representative channel width, the equations become:

$$\left.\begin{aligned} u(\partial u/\partial x)+v(\partial u/\partial y) &= -\partial p/\partial x - N(u+E_0)+R^{-1}\nabla^2 u,\\ u(\partial v/\partial x)+v(\partial v/\partial y) &= -\partial p/\partial y + R^{-1}\nabla^2 v,\\ 0 &= -\partial p/\partial z,\\ \partial u/\partial x + \partial v/\partial y &= 0, \end{aligned}\right\} \quad (2.7)$$

where $u = u_x/(Q/a)$, $v = u_y/(Q/a)$, $p = \tilde{p}/(\rho Q^2/a^2)$, $E_0 = E/(QB_0/a)$, $x = \tilde{x}/a$, $y = \tilde{y}/a$, $z = \tilde{z}/a$, $N = \sigma B_0^2 a^2/(\rho Q)$, and $R = \rho Q/\eta$. Also let $F_t(x) = f_t(\tilde{x})/a$ and $F_b(x) = f_b(\tilde{x})/a$.

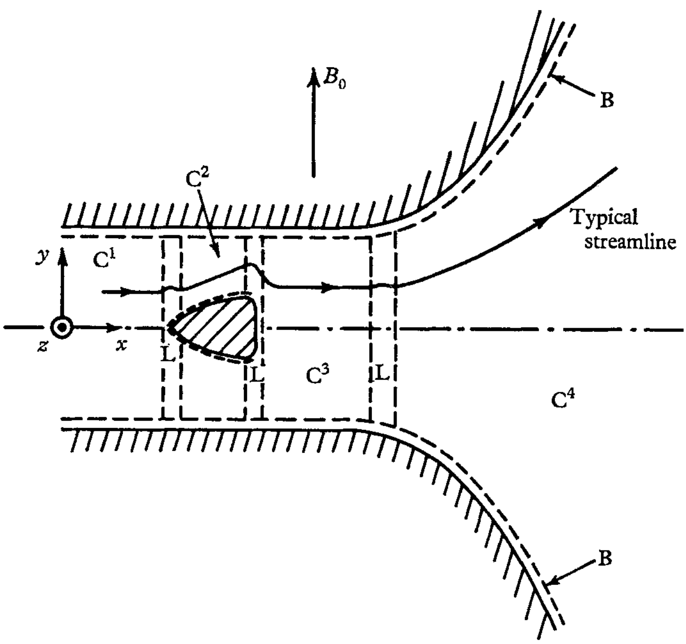

FIGURE 1. The various regions of flow, core (C), boundary layers (B), and Ludford layers (L), which are discussed in § 2.

To solve these equations and satisfy the boundary conditions we postulate the existence of various regions in the flow, which we examine in turn. The solutions which are found satisfy the boundary conditions, match each other at the boundaries of the regions, and are consistent with the original assumptions. We will now discuss the approximations to be used (see figure 1) by looking at the general problem of flow over a body placed in a duct with diverging walls.

Regions C (*core flow*)

In these regions, away from the boundaries, velocity gradients may be assumed to be $O(1)$ so that viscous forces are negligible and, since $N \gg 1$, the electromagnetic forces are very much greater than the inertial forces. Thus in these regions the electromagnetic force is balanced by the pressure gradient, and consequently the body force, $\mathbf{j} \times \mathbf{B}$, is irrotational. As pointed out by Shercliff (1965), the $\mathbf{j} \times \mathbf{B}$ force only affects the motion of an incompressible fluid with no free surfaces when it is *rotational*. It thus appears paradoxical that when no viscous effects are present, as the $\mathbf{j} \times \mathbf{B}$ force becomes sufficiently large it becomes *irrotational*. The explanation is that although in the final flow pattern the $\mathbf{j} \times \mathbf{B}$ force is irrotational, in the setting-up process the $\mathbf{j} \times \mathbf{B}$ force has to be *rotational*. Note that when the inertial forces are negligible the velocity is very simply determined by Ohm's law and the continuity equation, as shown in § 3. As Ludford has shown, this approximation breaks down in the region near a discontinuity in the slope of the boundary walls, as illustrated by the kink in the streamlines between C_1, C_2 and C_2, C_3 in figure 1.

Regions L (*Ludford layers*)

These regions emanate in the field direction from places where the slope of the boundary walls changes rapidly. Consequently v changes rapidly in the x-direction and therefore in these regions the inertial and viscous forces may be appreciable. These are the singular regions near the front and rear of a body which Ludford did not analyse. The structure of these regions, which we shall call 'Ludford' layers, is analysed in § 4 and is shown to depend on the relative size of M and R. For the parameter range of interest it is shown that the thickness of these layers is $O(N^{-\frac{1}{3}})$. Our analysis assumes that the slopes of the boundaries are always finite, although their rates of change may be infinite. This means we do not analyse the layers emanating from the rear of the body in figure 1, but only from the front. However, since duct walls usually have finite slopes, the analysis is valid for most practical situations.

Regions B

In these regions boundary layers are formed. We shall assume that their thickness is small compared with the size of the duct and that in these layers the dominant forces are viscous and electromagnetic. These assumptions are shown in § 5 to be equivalent to the conditions $N \gg 1$ and $M \gg 1$, the thickness of the boundary layer then being $O(M^{-1})$. In this analysis we implicitly assume that if N and M are sufficiently large there is no separation of the boundary layers, a supposition borne out by several experimental and theoretical investigations. In experimental investigations of the flow over cylinders, spheres, and flat plates (Tsinober 1963; Tsinober, Shtern & Shcherbinin 1963), and flow through a diverging channel (Heiser 1964), it has been shown that when the magnetic field is sufficiently great it can completely suppress the separation of a boundary layer. Some theoretical evidence for this phenomenon has been provided by Moreau (1964) who demonstrated that a transverse magnetic field can suppress the separation of boundary layers on a flat plate and on a cylinder.

3. Core flow

As Ludford (1961) has shown, in the limit $N \to \infty$, (2.7) reduce to

$$\partial p/\partial x + N(u+E_0) = 0, \tag{3.1 a}$$

$$\partial (p/N)/\partial y = 0, \tag{3.1 b}$$

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{3.1 c}$$

on allowing p to grow large with N and assuming that velocity gradients are $o(N)$. Equations (3.1) have the solution

$$u = -f'(x), \quad p = N[f(x) - xE_0], \quad v = yf''(x) + g(x). \tag{3.2}$$

Clearly, this solution cannot satisfy the no-slip condition $u = 0$, $v = 0$ at the walls. In fact, Hartmann layers of thickness $O(M^{-1})$ must form there, to reduce the tangential velocity of the core flow (3.2) to zero (see § 5). We therefore relax the no-slip condition, and require only that the normal velocity at the walls vanish.†

For flow in a duct, the top and bottom walls of which are described by the equations $y = F_t(x)$, $y = F_b(x)$ respectively, the boundary conditions are satisfied if

$$F_t f''(x) + g(x) = F_t'(x)\, u, \quad F_b f''(x) + g(x) = F_b'(x)\, u,$$

or

$$g(x) = \frac{F_b F_t' - F_t F_b'}{F_b - F_t}[-f'(x)]. \tag{3.3}$$

Furthermore, to satisfy the continuity requirement,

$$Q \int_{F_b}^{F_t} u\, dy = Q = (-f')(F_t - F_b)\, Q. \tag{3.4}$$

Thus,

$$u = \frac{1}{F_t - F_b}, \quad v = \frac{1}{(F_t - F_b)^2}[F_t'(y - F_b) + F_b'(F_t - y)], \tag{3.5}$$

and p may be found by integrating (3.1 a). If a body is placed in the duct, the top and bottom walls of which are at $y = C_t(x)$, $y = C_b(x)$, $(l_1 < x < l_2)$, then the solution when $l_1 < x < l_2$ for the flow between the body and the top wall is:

$$u = (F_t - C_t)^{-1}, \quad v = (F_t - C_t)^{-2}[F_t'(y - C_t) + C_t'(F_t - y)], \tag{3.6 a}$$

and for the flow between the body and the bottom wall,

$$u = (C_b - F_b)^{-1}, \quad v = (C_b - F_b)^{-2}[C_b'(y - F_b) + F_b'(C_b - y)]. \tag{3.6 b}$$

The solution for $x < l_1$, and $x > l_2$ is unaltered by the presence of the body. Thus the flow over a body in a duct is identical to the flow in two separate ducts, their walls being the top and bottom walls of the duct, the dividing streamlines and the top and bottom walls of the body. Therefore in the following analysis, where only flows in ducts are mentioned, we are implicitly treating flows over bodies as well.

† Ludford (1961) also deals with the solution (3.2). Since he is concerned with an infinite domain, however, he must take f' = constant, and cannot satisfy boundary conditions at infinity. These are satisfied by considering inertial effects for large y.

Our duct flow solution (3.5) (or pseudo duct flow (3.6)) holds whenever the wall slopes and curvatures are finite, since

$$\frac{1}{N}\frac{\partial v}{\partial x} = \frac{1}{N(F_t-F_b)^2}\left[F_t''(y-F_b)+F_b''(F_t-y)\right.$$
$$\left.-2\frac{F_t'-F_b'}{F_t-F_b}[F_t'(y-F_b)+F_b'(F_t-y)]\right],$$

so that if $F''_{t,b}$ or $F'^2_{t,b} = O(N)$, (3.1) fails to hold. As discussed in §2, we only deal with the case where $F'_{t,b} = O(1)$, so that we consider situations where (3.5) fails to hold owing to the curvature being $O(N)$. (The solution (3.6) always fails at the front and rear of a body except in the event of the body being cusp-shaped at these points.)

The solution (3.5) may be regarded as the leading terms in an asymptotic expansion:

$$u = u_0+\epsilon_1(N)\,u_1(x,y)+\epsilon_2(N)\,u_2+\dots,$$
$$v = v_0+\epsilon_1(N)\,v_1+\epsilon_2(N)\,v_2+\dots,$$
$$p = N[p_0+\epsilon_1(N)\,p_1+\dots],$$

where $\epsilon_{n+1} = o(\epsilon_n)$. The only choice of the sequence ϵ_n which leads to a set of equations for u_n, v_n, and p_n independent of N is $\epsilon_n = N^{-n}$. In considering the higher approximations, it is important to realize that the higher n the more the viscous terms become comparable to the electromagnetic and inertial terms, since the condition which must be satisfied for the viscous terms to be negligible in the nth approximation is that $M^{-2} \ll N^{-n}$.† Assuming this condition to be satisfied, the higher approximations may be found from

$$\frac{\partial u_n}{\partial y} = \sum_{k=0}^{n-1}(u_k\nabla^2 v_{n-1-k}-v_k\nabla^2 u_{n-1-k}), \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{3.7}$$

The arbitrary function of x arising from integration of (3.7) is fixed by requiring

$$\int_{F_b}^{F_t} u_n\,dy = 0 \quad \text{for} \quad n > 1,$$

and v_n follows from the formula

$$v_n = -\frac{\partial}{\partial x}\int_{F_b}^{y} u_n(x,\bar{y})\,d\bar{y}. \tag{3.8}$$

Note that v_n, as found from (3.8), satisfies the tangency conditions at the walls.

The solution for u_1 and v_1 is

$$\left.\begin{aligned} u_1 &= \left[\tfrac{1}{2}y^2-\frac{F_t^3-F_b^3}{6(F_t-F_b)}\right]\frac{dG_1}{dx}-\left[y-\frac{F_t+F_b}{2}\right]\frac{dG_2}{dx},\\ v_1 &= -\frac{\partial}{\partial x}\left\{\left[\frac{y^3-F_b^3}{6}-\frac{(F_t^3-F_b^3)(y-F_b)}{6(F_t-F_b)}\right]\frac{dG_1}{dx}-\left[\frac{y^2-F_b^2}{2}-\frac{(F_t+F_b)(y-F_b)}{2}\right]\frac{dG_2}{dx}\right\},\end{aligned}\right\} \tag{3.9}$$

where $\quad G_1 = \dfrac{1}{(F_t-F_b)^2}\dfrac{d}{dx}\left(\dfrac{F_t'-F_b'}{F_t-F_b}\right), \quad G_2 = \dfrac{1}{(F_t-F_b)^2}\dfrac{d}{dx}\left(\dfrac{F_tF_b'-F_bF_t'}{F_t-F_b}\right).$

† We are grateful to the referees for pointing this out to us. This condition is sufficient to allow one to ignore viscosity in the equations of motion. As pointed out in § 5, however, the Hartmann layer exerts a displacement effect on the core flow unless $M^{-1} \ll N^{-n}$.

Although we have indicated the method for finding higher approximations than the first, there is not much practical use in computing them. However, it is interesting to note that if the wall slope becomes $O(N^{\frac{1}{3}})$, then $N^{-n}u_n$ becomes of the same order as u_0 for all values of n. Therefore, to consider the first and higher approximations we must satisfy the condition $F'_t, F'_b \ll O(N^{\frac{1}{3}})$.

An example of the calculation of the zeroth and first approximations is given in § 6, where we consider the flow in converging and diverging ducts. The results are shown graphically in figure 5.

4. The Ludford layer equations

In regions where the wall curvature is $O(N)$ the inertial forces cannot be neglected and the solution for the core flow, (3.5), is no longer valid. Suppose such a region exists at $x = 0$, then we see from (3.5) that v_0 has an $O(1)$ jump while u_0 is

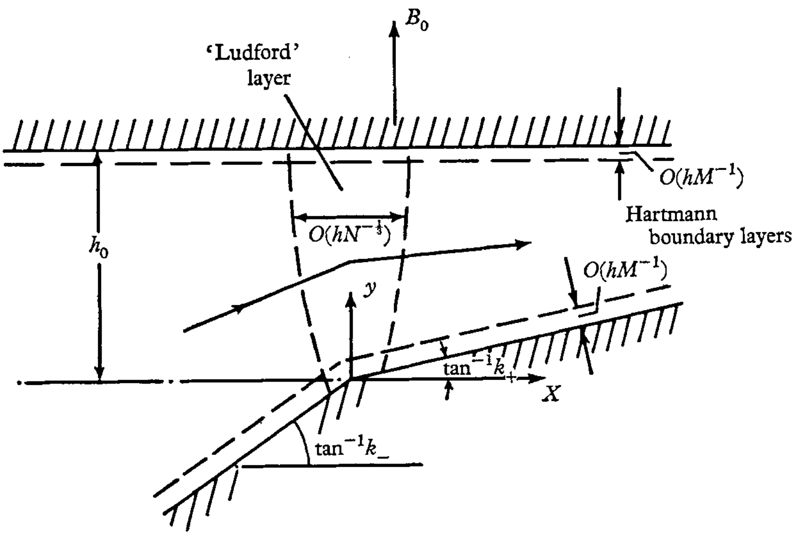

FIGURE 2. Illustration of the basic problem for the Ludford layer.

continuous at this point. (Note that v_1, u_1 and higher order terms are, in general, all discontinuous, as is shown by the example of § 6.) Now let us assume that the width of this region in the streamwise direction, δ, be very much less than the width of the channel, i.e. $\delta \ll 1$, and that the region appears to be a discontinuity in the limit $N \to \infty$. Then the problem is to show that such a layer can exist by finding a solution for u, v in the layer which matches u_0, v_0 in the core (see figure 2).

Stretch the x-co-ordinate by defining

$$X = x/\delta. \tag{4.1}$$

Since u is continuous in the core, it changes by an amount $O(\delta)$ in the layer. Accordingly, in the layer put

$$u = 1/h(0) + \delta(U(X, y)), \quad p = P/\delta, \tag{4.2, 3}$$

where $h(x) = F_t(x) - F_b(x)$ is the channel width at station x. Also let $h(0) = h_0$.

In terms of U, v, P, X, y, equations (2.7) are

$$\frac{1}{h_0}\frac{\partial U}{\partial X} + \delta U\frac{\partial U}{\partial X} + \delta v\frac{\partial U}{\partial y} = -\frac{1}{\delta^2}\frac{\partial P}{\partial X} - \delta NU - N\left(E_0 + \frac{1}{h_0}\right) + \frac{1}{R\delta}\frac{\partial^2 U}{\partial X^2}, \quad (4.4a)$$

$$\frac{1}{h_0}\frac{\partial v}{\partial X} + \delta U\frac{\partial v}{\partial X} + \delta v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{R\delta}\frac{\partial^2 v}{\partial X^2}, \quad (4.4b)$$

$$\frac{\partial U}{\partial X} + \frac{\partial v}{\partial y} = 0, \quad (4.4c)$$

where terms of $O(R^{-1})$ have been neglected compared to those of $O(\delta^{-2}R^{-1})$. All unknown quantities appearing in (4.4) and their derivatives are assumed to be $O(1)$. Equation (4.4a) is then

$$\partial P/\partial X = -\delta^3 NU - \delta^2 N(E_0 + h_0^{-1}) + O(\delta^2) + O(\delta R^{-1}), \quad (4.5)$$

while (4.4b) is

$$\frac{1}{h_0}\frac{\partial v}{\partial X} = -\frac{\partial P}{\partial y} + \frac{1}{R\delta}\frac{\partial^2 v}{\partial X^2} + O(\delta). \quad (4.6)$$

On eliminating the pressure and the core value of the $\mathbf{j}\times\mathbf{B}$ force, and ignoring the error terms, one obtains the equation

$$\frac{\partial^3 v}{\partial X^3} + h_0\delta^3 N\frac{\partial^2 v}{\partial y^2} - \frac{h_0}{R\delta}\frac{\partial^4 v}{\partial X^4} = 0, \quad (4.7)$$

which is also satisfied by U.

Depending on the values of N and R four possible situations may arise, leading to different values of δ, as listed below.

(a) *Electromagnetic-viscous balance*; $\delta^4 NR = \delta^4 M^2 = 1$, and $\delta R \ll 1$. Thus $\delta = M^{-\frac{1}{2}}$, and $M \gg R^2$ is the requirement for the existence of such a layer.

(b) *Inertial-viscous balance*; $\delta = R^{-1}$ and $\delta^3 N \ll 1$, which holds if

$$1 \ll R^{\frac{1}{2}} \ll M \ll R^2.$$

(c) *Inertial-viscous-electromagnetic balance*; $\delta^3 N = K^2 = O(1)$, $\delta = R^{-1}$, which holds if

$$M = KR^2.$$

(d) *Inertial-electromagnetic balance*; $\delta = (h_0/N)^{\frac{1}{3}} \ll 1$, $R\delta \gg 1$, which holds if

$$R^{\frac{1}{2}} \ll M \ll R^2.$$

We now concentrate on the type of layer which occurs when M, R and N have typical experimental values, e.g. $M = 500$, $R = 5000$, $N = 50$. Thus we can ignore situations (a) and (c), but we have to consider both the situations (b) and (d) since they may both occur in the same range of M and R. However, there is no solution to (4.7) which satisfies the required boundary conditions as $X \to \pm\infty$, if the electromagnetic term is neglected and a balance of the inertial and viscous forces is supposed to exist. Therefore we must consider the very much thicker layer which occurs in situation (d) where $\delta = O(N^{-\frac{1}{3}})$. We call this layer the Ludford layer in recognition of the similarities between this work and that of his 1961 paper. Its structure is governed by the equation

$$(\partial^3 v/\partial X^3) + h_0^2(\partial^2 v/\partial y^2) = 0. \quad (4.8)$$

We find that it is possible to construct a solution to this equation satisfying the boundary conditions and therefore we conclude that the errors, due to neglecting the higher order terms in (4.6) of $O(N^{-\frac{1}{3}})$, and due to neglecting the viscous terms of $O(M/R^2)^{\frac{2}{3}}$, do not affect the solution to this order of approximation.

It is important to note that, with this length scale, the boundary curvature still tends to infinity with N; in fact it is $O(N^{\frac{1}{3}})$ in (X, y)-space. Thus, the wall still has an abrupt change of slope at $X = 0$.

Since the problem is linear, we may break it into two parts. The core flow, from (3.5), may be written as $v_0 = v_1 + v_2$, where $v_1 = F_t'(y-F_b)/h^2$, $v_2 = F_b'(F_t-y)/h^2$, and separate solutions for the Ludford layer may be found which match with v_1 and v_2 as $X \to \pm\infty$. Thus, without loss of generality, we may assume that it is the bottom wall which curves abruptly, and consider only the problem of matching with v_2. Let $k_- = \lim_{x\to 0-} (F_b'(x)/h)$, $k_+ = \lim_{x\to 0+} (F_b'(x)/h)$ and put

$$F_t(0) - y = h(0)(1-Y) = h_0(1-Y).$$

The lower wall is then given by $Y = 0$ in the layer. The boundary conditions on $v(X, Y)$ in the layer are then

$$v(X, 1) = 0, \tag{4.9}$$

$$\left.\begin{aligned} v(X, 0) &= k_- \quad X < 0, \\ &= k_+ \quad X > 0, \end{aligned}\right\} \tag{4.10}$$

$$v \to k_-(1-Y) \quad \text{as} \quad X \to -\infty, \tag{4.11}$$

$$v \to k_+(1-Y) \quad \text{as} \quad X \to +\infty, \tag{4.12}$$

in order to match with v_2.

Structure of the layer

The problem for v may be further divided by writing

$$v = k_-(1-Y) + (k_+ - k_-)v^*, \tag{4.13}$$

where

$$\left.\begin{aligned} v^*(X, 1) &= 0, \\ v^*(X, 0) &= 1 \quad (X > 0), \\ &= 0 \quad (X < 0), \\ v^* &\to 1 - Y \quad \text{as} \quad X \to +\infty, \\ v^* &\to 0 \quad \text{as} \quad X \to -\infty, \end{aligned}\right\} \tag{4.14}$$

and

$$(\partial^3 v^*/\partial X^3) + (\partial^2 v^*/\partial Y^2) = 0. \tag{4.15}$$

The solution to (4.15) which satisfies the boundary conditions (4.14) is easily found by transform techniques, and may be written as

$$v^* = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\sin[(i\omega)^{\frac{3}{2}}(1-Y)]}{\sin(i\omega)^{\frac{3}{2}}} e^{i\omega X}\frac{d\omega}{\omega}. \tag{4.16}$$

In (4.16), the contour passes below the origin in the ω-plane. For $X > 0$ the integral may be evaluated by completing the contour in the upper half-plane, and the result is

$$v^* = 1 - Y - \frac{4}{3}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n\pi}\cos\left(\frac{(n\pi)^{\frac{2}{3}}3^{\frac{1}{2}}}{2}X\right)\exp\{-\tfrac{1}{2}(n\pi)^{\frac{2}{3}}X\}\sin[n\pi(1-Y)]\right\}. \quad (4.17)$$

When $X < 0$,

$$v^* = \frac{2}{3}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n\pi}\exp\{(n\pi)^{\frac{2}{3}}X\}\sin[n\pi(1-Y)]\right\}, \quad (4.18)$$

on completing the contour in the lower half-plane. Notice that

$$v^*(0, Y) = \tfrac{1}{3}(1-Y).$$

Graphs of v^* and $1-Y-v^*$ against X are plotted in figure 3; the discussion of the graphs is left to the conclusion.

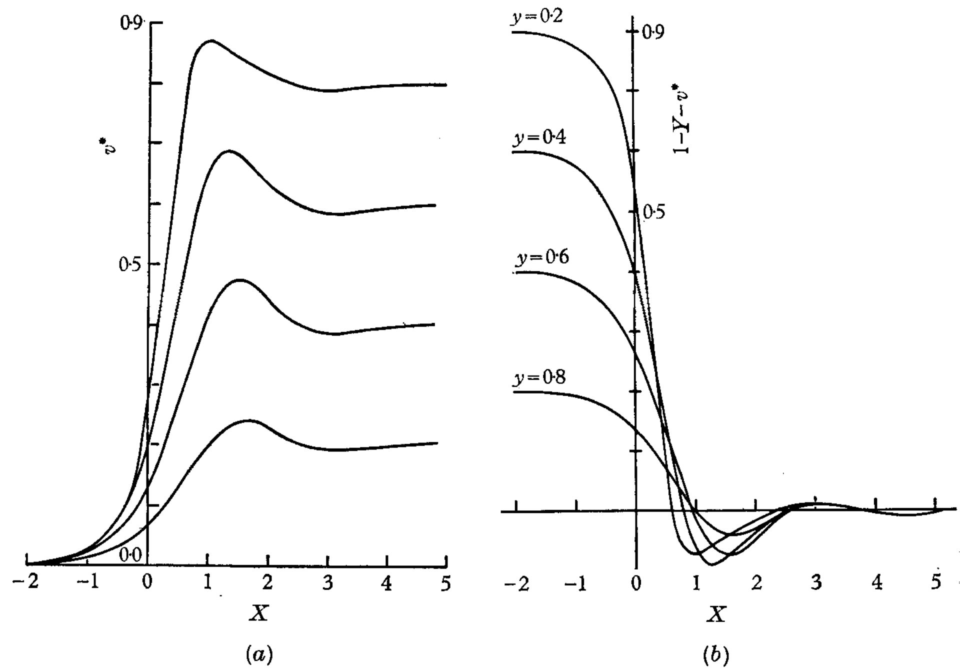

FIGURE 3. Profiles of v^* and $1-Y-v^*$ through the Ludford layers. These profiles indicate respectively the distribution of v when a straight duct joins a diverging duct and a converging duct joins a straight duct, as well as enabling the distribution of v in the general case to be calculated.

5. Hartmann boundary layers

We turn now to the boundary layers, B, and show how to calculate the flow there to the same order of accuracy as in the core. Consider the non-dimensional equation (3.1) written in terms of the s, n, z co-ordinates shown in figure 4, and let

$\overline{u}$ be parallel to the wall and $\overline{v}$ normal to it. The magnetic field is at an angle α to the normal of the wall. We have:

$$\overline{u}\frac{\partial \overline{u}}{\partial s}+\overline{v}\frac{\partial \overline{u}}{\partial n} = -\frac{\partial p}{\partial s}-N\cos\alpha(E_0+\overline{u}\cos\alpha-\overline{v}\sin\alpha)+\frac{1}{R}\nabla^2\overline{u}, \tag{5.1}$$

$$\overline{u}\frac{\partial \overline{v}}{\partial s}+\overline{v}\frac{\partial \overline{v}}{\partial n} = -\frac{\partial p}{\partial n}+N\sin\alpha(E_0+\overline{u}\cos\alpha-\overline{v}\sin\alpha)+\frac{1}{R}\nabla^2\overline{v}, \tag{5.2}$$

$$(\partial\overline{u}/\partial s)+(\partial\overline{v}/\partial n) = 0. \tag{5.3}$$

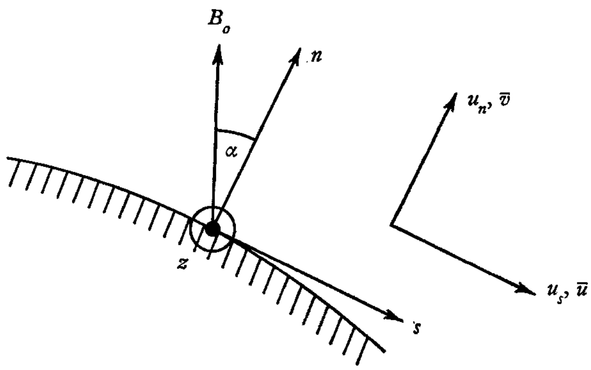

FIGURE 4. The notation for the boundary layer analysis in § 5.

These equations may be simplified by ignoring terms of order, δ_1, where $\delta_1(\ll 1)$ is the non-dimensional boundary-layer thickness. Then if we write $\zeta = n/\delta_1$, and $\overline{V} = \overline{v}/\delta_1$, the above equations become:

$$\overline{u}\frac{\partial \overline{u}}{\partial s}+\overline{V}\frac{\partial \overline{u}}{\partial \zeta} = -\frac{\partial p}{\partial s}-N\cos\alpha(E_0+\overline{u}\cos\alpha)+O(\delta_1 N)+\frac{1}{\delta_1^2 R}\frac{\partial^2\overline{u}}{\partial\zeta^2}, \tag{5.4}$$

$$\partial p/\partial\zeta = O(\delta_1 N)+O(R^{-1}), \tag{5.5}$$

$$(\partial\overline{u}/\partial s)+(\partial\overline{V}/\partial\zeta) = 0. \tag{5.6}$$

As with the Ludford layer, the structure of this boundary layer also depends on the relative sizes of M and R. In this case if $N(=M^2/R) \gg 1$ there is only one possible type of boundary layer, i.e. one in which the electromagnetic and viscous terms are very much greater than the inertial terms and balance each other. Hence it follows from (5.4) that

$$N = O(\delta_1^2 R)^{-1}.$$

or

$$\delta_1 = O(NR)^{-\frac{1}{2}} = O(M^{-1}). \tag{5.7}$$

For the boundary-layer thickness to be small compared with the duct width, δ must be small, or $M \gg 1$.

With these two approximations and using (5.7) we can obtain the zeroth-order solution for $\overline{u}$, $\overline{u}_0$, which satisfies

$$\left.\begin{aligned} O &= -N^{-1}(\partial p/\partial s)-\cos\alpha(E_0+\overline{u}_0\cos\alpha)+(\partial^2\overline{u}_0/\partial\zeta^2), \\ O &= -N^{-1}(\partial p/\partial\zeta), \end{aligned}\right\} \tag{5.8}$$

and the no-slip condition at the wall. The solution is

$$\bar{u}_0 = \bar{u}_{0\infty}(1-e^{-\zeta\cos\alpha}), \tag{5.9}$$

where $\bar{u}_{0\infty}$ is the component of the zeroth-order core velocity parallel to the wall (Stewartson 1960).

The higher order approximations to $\bar{u}$, $\bar{v}$ and p depend on the relative magnitude of M and N $(=M^2/R)$, since $\bar{u}$, $\bar{v}$ and p may be expressed as asymptotic expansions in M^{-1}, N^{-1}, or in a combination of powers of M^{-1} and N^{-1}. Let us consider the two limiting cases when $N^{-1} \gg M^{-1}$ and $M^{-1} \gg N^{-1}$. Then the expansions may be written:

$$\begin{aligned}
\bar{u} &= \bar{u}_0 + (N^{-1}\bar{u}_{1n} + N^{-2}\bar{u}_{2n} + \ldots) + [M^{-1}\bar{u}_{1m} + M^{-2}\bar{u}_{2m} + \ldots],\\
\bar{v} &= \bar{v}_0 + (N^{-1}\bar{v}_{1n} + N^{-2}\bar{v}_{2n} + \ldots) + [M^{-1}\bar{v}_{1m} + M^{-2}\bar{v}_{2m} + \ldots],\\
p &= N\{p_0 + (N^{-1}p_{1n} + N^{-2}p_{2n} + \ldots) + [M^{-1}p_{1m} + M^{-2}p_{2m} + \ldots]\},
\end{aligned}$$

where the expansion in either the square or round brackets vanish in the two cases. Then, in the first case, i.e. $N^{-1} \gg M^{-1}$, the expansion can only proceed until $N^{-r} \sim M^{-1}$ for some r, at which point it must either be terminated or a new mixed expansion of the form $N^{-r}M^{-s}$ must be considered. We may note that in this case M and R have the same relative magnitudes as in our analysis of the Ludford layer, i.e.

$$R^{\frac{1}{2}} \ll M \ll R$$

which is a condition satisfied in many experiments. Also in the first case it is important to realize that the higher order approximations may be matched to those in the core.

In the second case, i.e. $M^{-1} \gg N^{-1}$, or $M \gg R$, the expansion is carried out in terms of M^{-1}, or equivalently δ_1 which means that the core velocity is not regarded as parallel to the wall. Therefore the core velocity also has to be expressed as a series in δ_1 and has to be matched to the boundary-layer solution in such a way that the core velocity ceases to be independent of the boundary-layer flow. We ignore this expansion since it is of no practical use and concentrate on the first case.

We first find $\bar{u}_{1n}$, using the zeroth order solution (5.9). Now $\bar{u}_{1n}$ satisfies

$$\left.\begin{aligned}
\bar{u}_0\frac{\partial \bar{u}_0}{\partial s} + \frac{\partial \bar{u}_0}{\partial \zeta}\left(-\int_0^\zeta \frac{\partial \bar{u}_0}{\partial s}\,d\zeta\right) &= -\frac{\partial p_{1n}}{\partial s} - \bar{u}_{1n}\cos^2\alpha + \frac{\partial^2 \bar{u}_{1n}}{\partial \zeta^2},\\
0 &= -\frac{\partial p_{1n}}{\partial \zeta}.
\end{aligned}\right\} \tag{5.10}$$

Hence

$$\frac{\partial^2 \bar{u}_{1n}}{\partial \zeta^2} - \bar{u}_{1n}\cos^2\alpha = \bar{u}_{0\infty}\frac{d(\bar{u}_{0\infty})}{ds}\left[(1-e^{-\zeta\cos\alpha})^2 - \cos\alpha\, e^{-\zeta\cos\alpha}\left(\zeta - \frac{1-e^{-\zeta\cos\alpha}}{\cos\alpha}\right)\right] + \frac{\partial p_{1n}}{\partial s}. \tag{5.11}$$

Outside the layer $\bar{u}_{1n} = \bar{u}_{1\infty}$ and

$$\partial p_{1n}/\partial s = -\bar{u}_{1\infty}\cos^2\alpha - \bar{u}_{0\infty}(d\bar{u}_{0\infty}/ds).$$

Hence we can rewrite (5.11) as

$$\frac{\partial^2 \bar{u}_{1n}}{\partial \zeta^2} - \bar{u}_{1n}\cos^2\alpha = \frac{d(\bar{u}_{0\infty}^2/2)}{ds}[-(1+\zeta\cos\alpha)e^{-\zeta\cos\alpha}] - \bar{u}_{1\infty}\cos\alpha.$$

The solution for $\bar{u}_{1n}$ is

$$\bar{u}_{1n} = \bar{u}_{1\infty}(1-e^{-\zeta\cos\alpha}) + (d(\bar{u}^2_{0\infty})/ds)\left[\left(\frac{3\zeta+\zeta^2\cos^2\alpha}{8\cos\alpha}\right)e^{-\zeta\cos\alpha}\right].$$

To find $\bar{u}_{1\infty}$ and $\bar{u}_{0\infty}$ we use the results of §3, noting that

$$\bar{u}_{0\infty} = u_0\cos\alpha + v_0\sin\alpha,$$

and

$$\bar{u}_{1\infty} = u_1\cos\alpha + v_1\sin\alpha,$$

where $\tan\alpha = F'_b$ and u_0, v_0, u_1, v_1, are as defined in §3.

In principle, higher order terms in the expression for $\bar{u}$ may be found, since only linear equations need be solved. The algebra is complicated, however.

6. Example: flow through straight-walled converging and diverging ducts

We now consider the flow through a simple duct as an example. The expressions,

$$y = \pm 1 \quad \text{for} \quad x < 0 \quad \text{and} \quad y = \pm(1+x\tan\alpha) \quad \text{for} \quad x > 0,$$

represent a diverging duct if $\alpha > 0$. In these relations and those below which follow from it, if x is replaced throughout by $-x$, the solution for a converging duct is obtained.

Then the zeroth-order solution outside the Ludford layers is:

$x < 0$, core

$$\left.\begin{aligned} u_0 &= 1, \quad v_0 = 0, \\ \partial p_0/\partial x &= -N(1+E_0); \end{aligned}\right\} \tag{6.1}$$

boundary layer

$$\bar{u}_0 = 1-e^{-\zeta}; \tag{6.2}$$

$x > 0$, core

$$\left.\begin{aligned} u_0 &= \frac{1}{2(1+x\tan\alpha)}, \quad v_0 = \frac{y\tan\alpha}{2(1+x\tan\alpha)^2}, \\ \partial p_0/\partial x &= -N\left[\frac{1}{2(1+x\tan\alpha)} + E_0\right]; \end{aligned}\right\} \tag{6.3}$$

boundary layer

$$\bar{u}_0 = (1-e^{-\zeta})/(2\cos\alpha\,(1+x\tan\alpha)), \tag{6.4}$$

where

$$\begin{aligned} \zeta &= M(1-y), && \text{for } y > 0, \ x < 0; \\ &= M(1+y), && \text{for } y < 0, \ x < 0; \\ &= (1+x\tan\alpha - y)\,M\cos\alpha, && \text{for } y > 0, \ x > 0; \\ &= (1+x\tan\alpha - y)\,M\cos\alpha, && \text{for } y < 0, \ x > 0. \end{aligned}$$

When $x < 0$ the first-order solution is

$$u_1 = v_1 = \partial p_1/\partial x = 0.$$

(All higher orders are also zero.) When $x > 0$ the first-order solution is:

core

$$\left.\begin{aligned} u_1 &= \frac{\tan^3\alpha}{2(1+x\tan\alpha)^5}\left(y^2 - \frac{(1+x\tan\alpha)^2}{3}\right), \quad v_1 = \frac{y\tan^4\alpha}{2(1+x\tan\alpha)}\left[\frac{5y^2}{3} - (1+x\tan\alpha)^2\right], \\ \frac{\partial p_1}{\partial x} &= -\left\{u_0\frac{\partial u_0}{\partial x} + u_1\right\} = \frac{\tan\alpha}{(1+x\tan\alpha)^3}\left[\frac{1}{4} + \frac{\tan^2\alpha}{6} - \frac{y^2\tan^2\alpha}{2(1+x\tan\alpha)^2}\right]; \end{aligned}\right\} \tag{6.5}$$

boundary layer

$$\bar{u}_1 = \frac{\tan^3\alpha}{3\cos\alpha(1+x\tan\alpha)^3}\left\{1-\left(1+\left(\frac{9\cos\alpha}{16\sin^2\alpha}\right)\zeta+\left(\frac{3\cos^3\alpha}{16\sin^2\alpha}\right)\zeta^2\right)e^{-\zeta}\right\}. \quad (6.6)$$

In figure 5 we show velocity profiles in the core for flow in diverging and converging ducts, i.e. positive and negative α, and in figure 6 we show velocity profiles for the components of velocity parallel to the wall, $\bar{u}$, in the boundary layers.

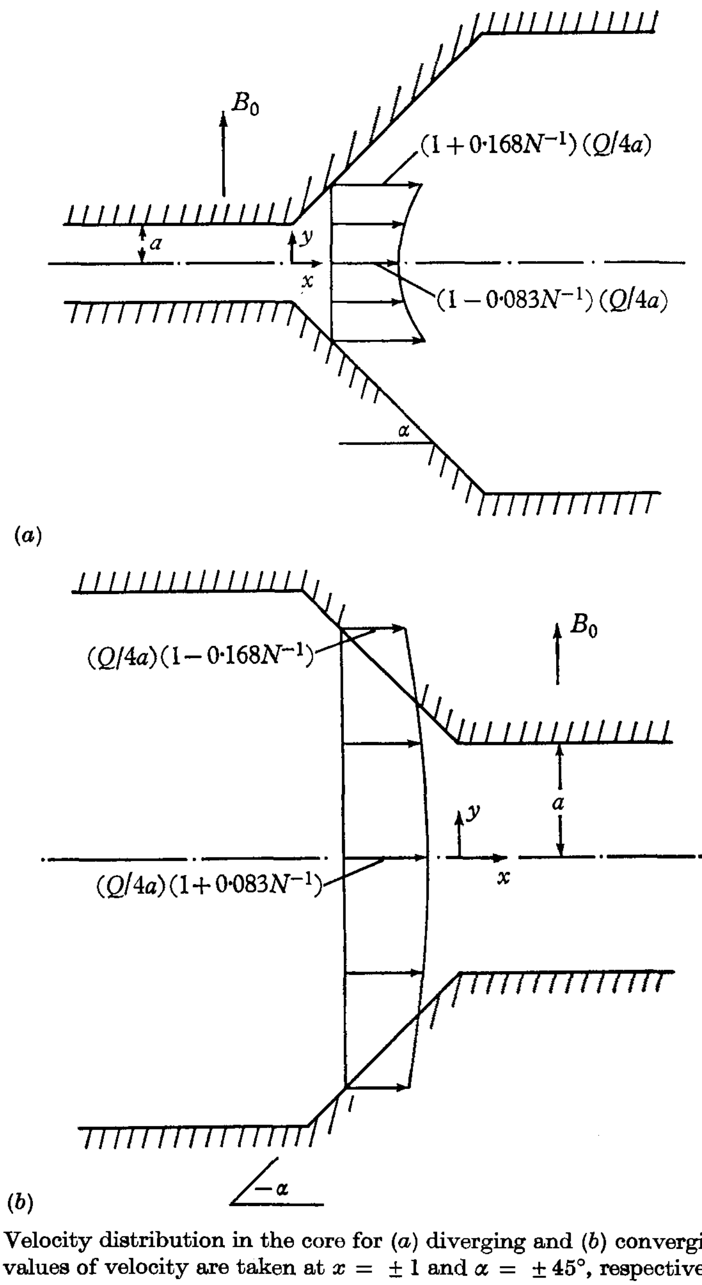

FIGURE 5. Velocity distribution in the core for (*a*) diverging and (*b*) converging flows. The values of velocity are taken at $x = \pm 1$ and $\alpha = \pm 45°$, respectively.

7. Discussion

(a) *Core and boundary layers*

The example presented in § 6 reveals some of the effects of considering higher order terms in the core and boundary-layer flows. Although the zeroth-order approximations for the core flow are identical in converging and diverging ducts (except for direction, of course), the first-order approximations differ in a surprising way. For a given value of x, the core velocity in a straight-walled diverging duct, such as that considered in § 6, is greatest near the walls and least in the centre, whereas for flow in a converging duct the reverse is true. There

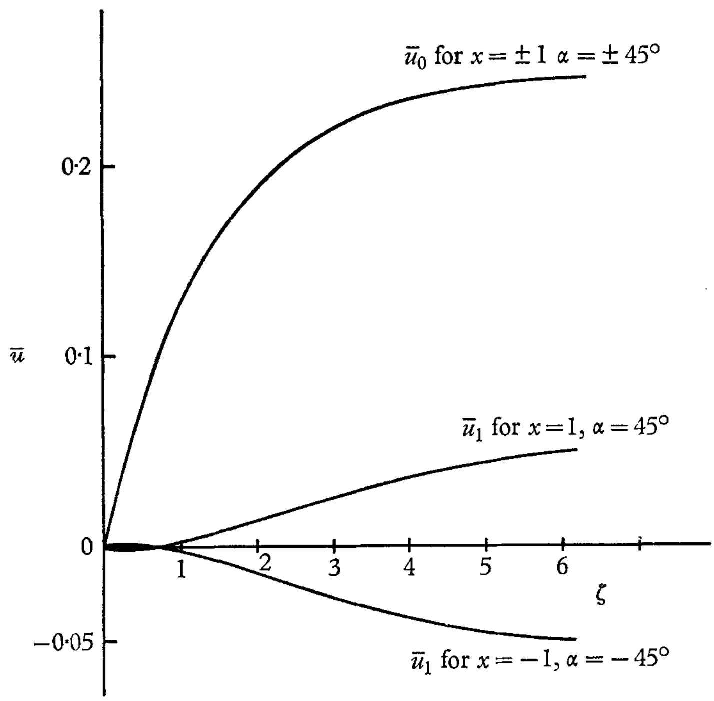

FIGURE 6. The velocity profiles of $\bar{u}_0$ and $\bar{u}_1$ in the boundary layer where $x = \pm 1$ and $\alpha = \pm 45°$. (Note $\bar{u} = \bar{u}_0 + N^{-1}\bar{u}_1$ and $\zeta = M^{-1}n$, where n is the co-ordinate normal to the wall.)

seems to be no obvious physical explanation for this effect, which only occurs in certain types of duct since, if the duct width is proportional to $1/(1-x)$ for $1 > x > 0$, the velocity is greatest in the centre for a *diverging* duct and least in a *converging* duct. Thus we conclude that the first, and presumably, higher approximations to the velocity profile are very sensitive to the rate of change of the duct width with distance along it.

It is of interest to compare the values of u_0 and u_1 in our example of § 6 in order to calculate the value of N which enables the required condition, $N^{-1}u_1 \ll u_0$, to be satisfied. For example, when $\alpha = 45°$, $x = 1$ and $y = 0$, $u_1/u_0 = \frac{1}{12}$, so that even if N is as low as 5 the conditions for the analysis of the core would be well satisfied. On the other hand, for the analysis of the Ludford layers we must satisfy the condition that $N^{\frac{1}{3}} \gg 1$ so that in an experiment where $N \simeq 10$, say, the experimental core flow would be adequately described by our theory but not the experimental Ludford layers.

Figure 6 indicates how inertial effects become apparent in the Hartmann boundary layers when the first-order approximation is considered. When the core flow is decelerating as in a diverging duct, there is a slight tendency for back flow to develop near the wall, whereas, when the core flow is accelerating, the flow near the wall is faster. It is interesting that the tendency for back flow to develop in a diverging duct is very much *greater* when $\overline{u}_{1\infty} < 0$, as in a duct whose width is proportional to $1/(1-x)$, than when $\overline{u}_{1\infty} > 0$, as in the example of § 6, which indicates that the first-order approximation of the core flow has an important effect on the boundary-layer flow.

(*b*) *Ludford layers and the relation between Ludford's solution and the duct flow problems*

We have considered the structure of the Ludford layer when the core flow is continuous in u_0 and when the predominant forces are pressure, inertial, and electromagnetic. The criteria to be satisfied by M and R for our analysis are

$$R^2 \gg M \gg R^{\frac{1}{2}}. \tag{7.1}$$

In this case the thickness of the layer, δ, is $O(N^{-\frac{1}{3}})$. The key to a physical understanding of the layer lies in the role of the pressure gradients; the pressure gradient in the y-direction, $\partial p/\partial y$, is $O(N^{-1})$ in the core, while in the Ludford layer it is $O(N^{\frac{1}{3}})$, because it is the pressure gradient which accelerates the fluid in the y-direction, *not*, of course, the electromagnetic force. Since the pressure varies in the y-direction, there must be a component of $\partial p/\partial x$ of $O(N^{\frac{2}{3}})$ which also varies in the y-direction, i.e. different from the core value of $\partial p/\partial x = O(N)$, and this secondary component of $\partial p/\partial x$ is balanced by the $\mathbf{j} \times \mathbf{B}$ force produced by a perturbation velocity U of $O(N^{-\frac{1}{3}})$. The practical significance of the pressure gradient is that, since pressures are measured more easily than velocities, probably the best way to confirm the existence of Ludford layers is to check whether the pressure difference across an *asymmetric* channel at a point where the wall slope changes suddenly is $O(N^{\frac{1}{3}})$.

Note that the graphs of v^* and $(1-Y-v^*)$ shown in figure 3 can be interpreted directly since v^* $(=v/k_+)$ is proportional to v when $k_- = 0$, that is, a straight duct joining a diverging duct. Also, $(1-Y-v^*)$ $(=v/k_-)$ is proportional to v when $k_+ = 0$, that is, for a converging duct joining a straight duct. From v^* and $(1-Y-v^*)$ we can calculate v for the general case in which k_+ and k_- are both non-zero. Also note that the damped wave, for which there is no obvious explanation, always occurs downstream of any change in the slope of the duct wall.

Our original solution for v in the Ludford layer, not reported here, suggested to us that a uniform approximation for the unconfined flow over a body, valid for both Ludford's inner and outer regions, is given by

$$v = -\int_{-1}^{+1} F'(t)\frac{\partial}{\partial t}\mathscr{H}\left(\frac{N^{\frac{1}{3}}(x-t)}{[y-F(x)]^{\frac{2}{3}}}\right)dt,\dagger \tag{7.2}$$

where $y = \pm F(x)$, $(-1 < x < 1)$, is the profile of the body. This is a slight modification of Ludford's solution, y in his being replaced here by $(y-F(x))$. The modification is proposed only for sharp nosed, symmetric bodies, and was in fact hinted at in Ludford's (1961) paper. In the appendix, it is shown that (7.2) reduces to Ludford's outer solution when $y = O(N^{\frac{1}{2}})$, and his inner solution when $y = O(1)$, with corrections which amount to placing Ludford layers at the nose and tail.

On the surface, it appears as though there is a discrepancy between our results and Ludford's for bodies in ducts as the duct width, a, tends to infinity. For example, his analysis predicts that a non-symmetric body disturbs the flow upstream, whereas ours indicates that for such a body in a straight-walled duct there is no upstream effect even as $a \to \infty$, as shown in (3.6). However, our analysis is in fact not appropriate for studying the effect of flow past bodies whose characteristic dimension is very small compared to a, and Ludford's solution must be appealed to in this situation.

(c) *Limitations of the analysis*

Our analysis has been for two-dimensional flows, but since experiments have to be performed in finite-sized ducts the effects of the side walls parallel to the fields must be considered. Also, it is only by considering the side walls that we can determine E_z. These walls may be non-conducting, or, if conducting, they may be split up into segments. They may also diverge in the z-direction. In these cases E_z may vary in the x-direction and E_x is likely to be non-zero, in which case secondary flows may result, and our analysis will not hold except perhaps in the centre of the duct away from the side walls. Therefore, our analysis is expected to be most applicable in a duct with continuous conducting walls parallel to B_0 since then E_z will be uniform in the core and $E_x = 0$. E_z will then be determined by considering the external electrical circuit and the total current leaving the duct. Even in this case the analysis will fail where the conducting-electrode walls end at the edge of the power extraction or injection region. However, we hope to extend this work to cover these more complicated situations.

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† Ludford's function $\mathscr{H}(\eta)$ is defined by

$$\mathscr{H}(\eta) = 1-\frac{1}{3\pi i}\int_{c-i\infty}^{c+i\infty}\exp(\lambda y-\lambda^{\frac{2}{3}}x)\frac{d\lambda}{\lambda} \quad (c > 0).$$

Notice that it is a function of the similarity variable $\eta = x/y^{\frac{3}{2}}$. See his 1961 paper for details.

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Appendix

In the appendix, we show that (7.2) is a uniform approximation to Ludford's problem (in the sense indicated).

For $y = O(N^{\frac{1}{2}})$, put $y = ZN^{\frac{1}{2}}$ into (7.2). Then

$$\mathscr{H}\left(\frac{(x-t)\,N^{\frac{1}{3}}}{[y-F(x)]^{\frac{2}{3}}}\right) = \mathscr{H}\left[\frac{x-t}{(Z-N^{-\frac{1}{2}}F)^{\frac{2}{3}}}\right] = \mathscr{H}\left(\frac{x-t}{Z^{\frac{2}{3}}}\right) + O(N^{-\frac{1}{2}})$$

for Z bounded away from zero, by Taylor's theorem. Thus (7.2) agrees with Ludford's result to $O(N^{-\frac{1}{2}})$ in the outer region.

To consider the situation in the inner flow, when $y = O(1)$, first integrate (7.2) by parts to get

$$v = F'(-1)\,\mathscr{H}\left[\frac{x+1}{(y-F)^{\frac{2}{3}}}\,N^{\frac{1}{3}}\right] - F'(1)\,\mathscr{H}\left[\frac{N^{\frac{1}{3}}(x-1)}{(y-F)^{\frac{2}{3}}}\right] + \int_{-1}^{1} F''(t)\,\mathscr{H}\,dt.$$

Now, for $-1 < x < 1$,

$$\int_{-1}^{1} F''\mathscr{H}\,dt = \int_{-1}^{x-\epsilon} F''\mathscr{H}\,dt + \int_{x+\epsilon}^{1} F''\mathscr{H}\,dt + O(\epsilon),$$

where ϵ is a small number, but $N^{\frac{1}{3}}\epsilon \gg 1$. Using Ludford's asymptotic formula for large η, i.e.

$$\mathscr{H}(\eta) = O(|\eta|^{-\frac{3}{2}}) \quad (\eta < 0),$$
$$\mathscr{H}(\eta) = 1 + O[\eta^{-3}\exp(-\tfrac{4}{27}\eta^3)] \quad (\eta > 0),$$

we have

$$\int_{-1}^{1} F''\mathscr{H}\,dt = \int_{-1}^{x-\epsilon} F''(t)\,[1 + O\{[(y-F)^2/N\epsilon^3]\exp\{-4N\epsilon^3/27(y-F)^2\}]\,dt$$
$$+ O\{(y-F)(N\epsilon^3)^{-\frac{1}{2}}\} + O(\epsilon).$$

Thus,

$$v = F'(x) + F(-1)\left[\mathscr{H}\left(\frac{N^{\frac{1}{3}}(x+1)}{[y-F(x)]^{\frac{2}{3}}} - 1\right) - F'(1)\mathscr{H}\left[\frac{N^{\frac{1}{3}}(x-1)}{(y-F)^{\frac{2}{3}}}\right]\right.$$
$$+ O\{(y-F)(N\epsilon^3)^{-\frac{1}{2}}\} + O(\epsilon), \quad \text{(A 1)}$$

so that if the point x considered is not within a distance $O(N^{-\frac{1}{3}})$ from either the leading or trailing edge,

$$v = F'(x) + O\{(y-F)(N\epsilon^3)^{-\frac{1}{2}}\} + O(\epsilon).$$

Providing ϵ is chosen so that $\epsilon^3 N \to \infty$ as $N \to \infty$, this reduces to Ludford's limit solution $v = F'(x)$ in $-1 < x < 1$.

On the other hand, if the point in question approaches either -1 or $+1$ to $O(N^{-\frac{1}{3}})$, then one of the extra terms in (A 1) remains. For example, near $x = -1$, the solution is

$$v \sim F'(-1)\mathscr{H}(N^{\frac{1}{3}}(x+1)/[y-F(x)]^{\frac{2}{3}}).$$

Thus the fluid is taken smoothly through the Ludford layers at the leading and trailing edges, and the discontinuity in Ludford's inner solution is removed.

So far the presence of $F(x)$ in the argument of $\mathscr{H}(\ldots)$ has not been utilized. It is inserted there in order to satisfy the boundary condition on v at $y = F(x)$. In this regard, note that all of the error terms in the above vanish as $y \to F(x)$ and the results are then exact, i.e. $v = F'(x)$, as required.

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